Unit 1 · Speaking Mathematically Cheat Sheet

Unit 1 focus: statements, variables, universal and existential statements, conditional statements

1 KEY VOCABULARY

statement	True or false, but not both.	"17 is prime" (true)
open sentence	Truth depends on a variable.	$x + 2 = 5$
domain	The set the variable comes from.	$D = \{1, 2, 3\}$
truth set	All values in D that make it true.	$x^2 = 4$ in $\mathbb{Z}$: $\{-2, 2\}$
counterexample	Makes a "for all" claim false.	2 is an even prime

2 STATEMENT OR NOT?

Sentence	Statement?
12 is odd.	**Yes** (false statement)
Is 91 prime?	No: a question
$x + 5 = 9$	No: depends on x
For every real x, $x + 0 = x$.	**Yes** (true): x is quantified

A false sentence is still a statement. Quantifying the variable makes an open sentence a statement.

3 UNIVERSAL AND EXISTENTIAL

Form	To prove / to disprove
$\forall x \in D,\ P(x)$ "for all"	prove: general argument · disprove: **one counterexample**
$\exists x \in D,\ P(x)$ "there is"	prove: **one example** · disprove: general argument
Negations	$\neg\forall x\, P(x) \equiv \exists x\, \neg P(x)$ and $\neg\exists x\, P(x) \equiv \forall x\, \neg P(x)$

Order matters: $\forall x\, \exists y\, (x + y = 0)$ is true, but $\exists y\, \forall x\, (x + y = 0)$ is false.

4 TRUTH SET EXAMPLE

Test every element of the stated domain.

Example: D = {0, 1, 2, 3, 4, 5, 6}; find the truth set of $n^2 - 6n + 8 = 0$.
Factor: $(n - 2)(n - 4) = 0$
Check D: only 2 and 4 work

Answer: $\{2, 4\}$

5 CONDITIONAL STATEMENTS

Words	Means $p \to q$ with
if p, then q · q whenever p	hypothesis p, conclusion q
p only if q	"only if" introduces the **conclusion**
q is necessary for p	$p \to q$
p is sufficient for q	$p \to q$
q unless r	$\neg r \to q$

$p \to q$ is false **only** when p is true and q is false. A false hypothesis makes it (vacuously) true.

6 COMMON MISTAKES

✓ Do remember

- Name the domain before deciding truth.
- One counterexample disproves "for all".
- Examples never prove "for all" on an infinite domain.

✗ Common mistakes

- Negating "all" as "none": it becomes "some ... not"
- Reading "p only if q" as $q \to p$: it is $p \to q$
- Calling a false sentence "not a statement": false statements count

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More practice: set notation, subsets, Cartesian products, ordered pairs and strings

1 SET NOTATION

Notation	Meaning and example
roster	$\{2,4,6,8\}$: list the elements
set-builder	$\{n \in \mathbb{Z} \mid 0 < n < 4\} = \{1,2,3\}$
$x \in A$	x is an element of A
$\varnothing$	the empty set, no elements
$\lvert A \rvert$	number of elements
$\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$	integers, rationals, reals

Order and repetition never matter: $\{1,2,2\} = \{2,1\}$.

2 ELEMENT (∈) VS SUBSET (⊆)

$2 \in \{1,2,3\}$	**true**
$\{2\} \in \{1,2,3\}$	**false**
$\{2\} \subseteq \{1,2,3\}$	**true**
$\varnothing \subseteq A$ for every A	**true**
$\{1,2\} \in \{\{1,2\},3\}$	**true**
$\lvert\{\varnothing\}\rvert$	**1**

3 COUNTING SUBSETS

Each element is either in or out of a subset, so a set with n elements has 2^n subsets ($2^n - 1$ proper subsets).

Example: How many subsets of $\{1,2,3,4,5,6\}$ contain 1 but not 2?
Fixed: 1 in, 2 out
Free: 3, 4, 5, 6 → 2^4

Answer: 16

4 CARTESIAN PRODUCTS

$A \times B = \{(a,b) \mid a \in A,\ b \in B\}$ and $\lvert A \times B \rvert = \lvert A \rvert \cdot \lvert B \rvert$.
Ordered pairs: $(a,b) = (c,d)$ only if $a = c$ and $b = d$.

Example: $A = \{1,2\}$, $B = \{x,y,z\}$
List: (1, x), (1, y), (1, z), (2, x), (2, y), (2, z)
Note: $A \times B \neq B \times A$ here

Answer: 6 ordered pairs

5 STRINGS

Idea	Example
string over an alphabet Σ	0110 over $\{0,1\}$, length 4
empty string λ	length 0
concatenation st	“ab”·“ba” = “abba”
strings of length k over n symbols	n^k, e.g. $2^5 = 32$ bit strings
at least one 1 (length 6)	$2^6 - 1 = 63$

6 CHALLENGE

Quantifier order. Let $P(x,y)$: “$y > x$” on $\mathbb{Z}$.
• $\forall x\, \exists y\, P(x,y)$: true (take $y = x + 1$).
• $\exists y\, \forall x\, P(x,y)$: false (no integer is larger than every integer).
Negate the first: $\exists x\, \forall y\, (y \leq x)$ — “some integer is the largest”, which is false, as a negation of a true statement must be.