Unit 4 · Sequences, Induction & Recursion Cheat Sheet

Unit 4 focus: explicit and recursive sequences, arithmetic and geometric patterns, factorials, binomial coefficients, Fibonacci

1 SEQUENCE LANGUAGE

explicit formula	term from n directly	$a_n = 3n - 1$: 2, 5, 8, …
recursive rule	term from earlier terms	$a_1 = 2,\ a_n = a_{n-1} + 3$
summation	add the terms	$\sum_{k=1}^{4} k^2 = 30$
product	multiply the terms	$\prod_{k=1}^{4} k = 24$

2 ARITHMETIC AND GEOMETRIC

	Arithmetic	Geometric
rule	add d	multiply by r
nth term	$a_n = a_1 + (n-1)d$	$a_n = a_1 r^{n-1}$
sum of n terms	$\frac{n(a_1+a_n)}{2}$	$a_1 \frac{r^n-1}{r-1}$
example	4, 7, 10, …: $a_{10} = 31$	3, 6, 12, …: $a_6 = 96$

$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$, so $1 + \cdots + 100 = 5050$.

3 FACTORIALS AND C(N, R)

$n! = n(n-1)\cdots 1$, $0! = 1$. $\binom{n}{r} = \frac{n!}{r!\,(n-r)!}$ counts r-element subsets. Pascal: $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$.

Example: Evaluate $\binom{7}{3}$.
Cancel: $\frac{7\cdot 6\cdot 5}{3\cdot 2\cdot 1}$
Pascal check: $\binom{6}{2} + \binom{6}{3} = 15 + 20$

Answer: 35

4 FIBONACCI NUMBERS

n	1	2	3	4	5	6	7	8	9	10
F_n	1	1	2	3	5	8	13	21	34	55

$F_1 = F_2 = 1$, $F_n = F_{n-1} + F_{n-2}$. Sum: $F_1 + \cdots + F_n = F_{n+2} - 1$. Ratios F_{n+1}/F_n approach $\varphi \approx 1.618$.

5 ITERATION

Unwind the recurrence until a pattern appears, then check it.

Example: $m_1 = 1,\ m_k = 2m_{k-1} + 1$ (Tower of Hanoi)
Terms: 1, 3, 7, 15, 31
Pattern: $m_k = 2^k - 1$
Check: $2(2^{k-1} - 1) + 1 = 2^k - 1$

Answer: $m_{10} = 1023$

6 COMMON MISTAKES

✔ Do remember

- Check where the index starts (a_0 or a_1).
- Use the previous term, not the first term, in a recursion.
- Cancel factorials before multiplying.

✘ Common mistakes

- $a_{10} = a_1 + 10d$: it is $a_1 + 9d$
- $0! = 0$: $0! = 1$
- $\frac{8!}{6!} = (\frac{8}{6})!$: $8 \cdot 7 = 56$

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More practice: iteration, the principle of mathematical induction, writing inductive proofs, patterns and tessellations

1 MATHEMATICAL INDUCTION

Step	What you write
1. Base case	Check $P(a)$ for the first value a.
2. Inductive hypothesis	Assume $P(k)$ for an arbitrary $k \geq a$.
3. Inductive step	Show $P(k+1)$ follows from $P(k)$.
4. Conclusion	By induction, $P(n)$ is true for all $n \geq a$.

Dominoes: the first falls, and each one knocks over the next. Strong induction assumes $P(a), \ldots, P(k)$.

2 SUM FORMULA PROOF

Prove $1 + 3 + \cdots + (2n-1) = n^2$ for $n \geq 1$.

Example: Base $n = 1$: $1 = 1^2$.
Assume: $1 + \cdots + (2k-1) = k^2$
Add next term: $k^2 + (2k+1)$
Factor: $= (k+1)^2$

Answer: $P(k+1)$ **holds**

3 DIVISIBILITY PROOF

Rewrite the $k+1$ expression to contain the old one.

Example: Prove $4 \mid (5^n - 1)$ for $n \geq 0$.
Base: $5^0 - 1 = 0$
Step: $5^{k+1} - 1 = 5(5^k - 1) + 4$
Both parts: divisible by 4

Answer: true for all $n \geq 0$

4 INEQUALITY PROOF

Use the hypothesis, then a simple bound.

Example: Prove $2^n \geq n + 1$ for $n \geq 0$.
Base: $2^0 = 1 \geq 1$
Step: $2^{k+1} = 2 \cdot 2^k \geq 2(k+1) = 2k + 2$
Bound: $2k + 2 \geq k + 2$

Answer: $2^{k+1} \geq (k+1) + 1$

5 PATTERNS AND TESSELLATIONS

Fact	Example
regular n-gon interior angle $\frac{(n-2)180^\circ}{n}$	hexagon: 120°
diagonals of an n-gon $\frac{n(n-3)}{2}$	octagon: 20
triangular numbers $T_n = \frac{n(n+1)}{2}$	1, 3, 6, 10, 15
regular tessellations	only triangles (6 at a vertex), squares (4), hexagons (3)

A regular polygon tiles the plane alone exactly when its interior angle divides 360°.

6 MISTAKES AND CHALLENGE

✓ Do remember

- Always write the base case.
- In the step, start from $P(k)$; the goal is $P(k+1)$.
- Check the step works for every $k \geq$ the base value.

✘ Common mistakes

- Assuming $P(k+1)$: that is circular
- Base case at the wrong n: use the smallest claimed value
- Challenge: “all horses are one color”: the step fails from $k = 1$ to 2