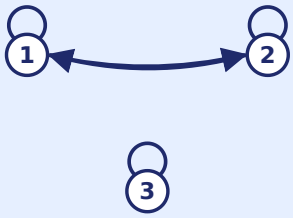
Unit 6 · Properties of Relations Cheat Sheet

Unit 6 focus: relations and inverses, directed graphs, reflexive, symmetric and transitive properties

1 RELATION VOCABULARY

relation on A	a subset of $A \times A$	$\{(1,2),(2,3)\}$
a R b	$(a,b) \in R$	$2 < 5$
inverse R^{-1}	reverse every pair	"taller than" → "shorter than"
digraph	arrow $a \to b$ for (a,b), loop for (a,a)	see the next panel

2 DIRECTED GRAPH OF R



$A = \{1,2,3\}$, $R = \{(1,1),(2,2),(3,3),(1,2),(2,1)\}$. Loops at every vertex: reflexive. Every arrow has a return arrow: symmetric. Every two-step path has a shortcut: transitive.

3 THE THREE PROPERTIES

Property	Definition (on set A)	Digraph test
reflexive	$(a,a) \in R$ for every $a \in A$	loop at every vertex
symmetric	$(a,b) \in R \Rightarrow (b,a) \in R$	arrows come in pairs
transitive	$(a,b),(b,c) \in R \Rightarrow (a,c) \in R$	every 2-step path has a shortcut

To **disprove** a property, give one failing pair (or pair of pairs).

4 CHECKING AN EXAMPLE

$A = \{1,2,3\}$, $R = \{(1,1),(2,2),(1,2),(2,3)\}$.

Example: Which properties fail?
Reflexive: no, $(3,3) \notin R$
Symmetric: no, $(1,2)$ but not $(2,1)$
Transitive: no, $(1,2),(2,3)$ but not $(1,3)$

Answer: none of the three hold

5 FAMILIAR RELATIONS

Relation	Refl.	Sym.	Trans.
$=$ on $\mathbb{R}$	✔	✔	✔
$<$ on $\mathbb{R}$	✘	✘	✔
$\leq$ on $\mathbb{R}$	✔	✘	✔
$\neq$ on $\mathbb{Z}$	✘	✔	✘
$a \mid b$ on $\mathbb{Z}^+$	✔	✘	✔
$\lvert x-y\rvert < 1$ on $\mathbb{R}$	✔	✔	✘

$|x-y| < 1$ fails transitivity: $0, 0.6, 1.2$.

6 COMMON MISTAKES

✔ Do remember

- Reflexive needs a loop at **every** element.
- Check every chain for transitivity, including ones that return to the start.
- The empty relation is symmetric and transitive (vacuously).

✘ Common mistakes

- R^{-1} = negate the numbers: reverse each pair
- One working example proves a property: you need every case
- Symmetric + transitive $\Longrightarrow$ reflexive: false if some element is related to nothing

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More practice: relations on infinite sets, partitions, equivalence relations and equivalence classes

1 COUNTING RELATIONS

On a set with n elements	Number
all relations	2^{n^2} ($n=3$: 512)
reflexive	2^{n^2-n} (64)
symmetric	$2^{n(n+1)/2}$ (64)
reflexive and symmetric	$2^{n(n-1)/2}$ (8)

2 PARTITIONS

A partition of A is a set of **nonempty**, **disjoint** blocks whose union is A. Its induced relation: $x\,R\,y$ when x, y are in the same block.

Example: Partition $\{\{1,2,3\},\{4,5\}\}$ of $\{1,\ldots,5\}$
Pairs: a block of size k gives k^2 pairs
Count: 3^2+2^2

Answer: 13 ordered pairs

3 EQUIVALENCE RELATIONS

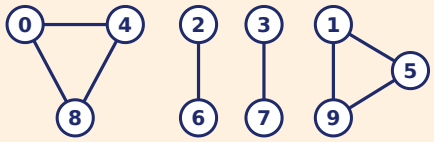
An **equivalence relation** is reflexive, symmetric and transitive. Examples: "same birthday", congruence $a \equiv b \pmod n$ ($n \mid a-b$), $x^2=y^2$. Not: "$\leq$" (not symmetric), "friend of" (not transitive).
Theorem: equivalence relations and partitions correspond one-to-one: the classes form a partition.

4 EQUIVALENCE CLASSES

$[a]=\{x \mid a\,R\,x\}$. Any two classes are equal or disjoint.

Example: Congruence mod 4 on $\{0,1,\ldots,9\}$
$[0]$**:** $\{0,4,8\}$
$[1]=[5]$**:** $\{1,5,9\}$
Others: $\{2,6\}$, $\{3,7\}$

Answer: 4 distinct classes



5 PROVING AN EQUIVALENCE RELATION

- **Relation:** $m\,R\,n \iff 3 \mid (m-n)$ on $\mathbb{Z}$.
- **Reflexive:** $m-m=0=3\cdot 0$.
- **Symmetric:** $m-n=3k \Rightarrow n-m=3(-k)$.
- **Transitive:** $m-n=3k$, $n-p=3j \Longrightarrow m-p=3(k+j)$.
- **Classes:** $[0],[1],[2]$ (remainders mod 3).

6 CHALLENGE

How many equivalence relations are there on $\{1,2,3,4\}$? Count partitions: one block (1), two blocks (7), three blocks (6), four blocks (1): **15** (the Bell number B_4).
On $\mathbb{Z}\times(\mathbb{Z}-\{0\})$, $(a,b)\,R\,(c,d) \iff ad=bc$ is an equivalence relation whose classes are the rational numbers.