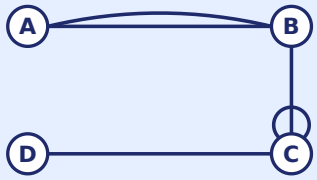
Unit 8 · Graphs and Trees Cheat Sheet

Unit 8 focus: vertices, edges, loops, degrees, directed graphs, constructing graphs and graph coloring

1 GRAPH VOCABULARY

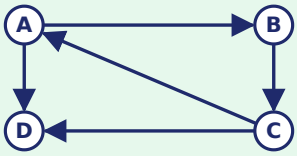
vertex, edge	points and the connections between them	K_4 has 4 vertices, 6 edges
loop	an edge from a vertex to itself	adds 2 to the degree
parallel edges	two edges with the same endpoints	A–B twice
simple graph	no loops, no parallel edges	K_5
degree	number of edge-ends at a vertex	see the figure

2 DEGREES AND THE HANDSHAKE THEOREM



Degrees: A 2, B 3, C 4 (the loop counts 2), D 1. Sum $= 10 = 2 \cdot 5$ edges. **Handshake:** $\sum \deg(v) = 2|E|$, so the number of odd-degree vertices is even.

3 DIRECTED GRAPHS



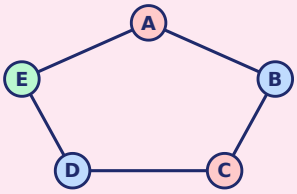
In-degree = arrows in; out-degree = arrows out. $\sum \text{in} = \sum \text{out} = |E|$. Here A: in 1, out 2. Use a directed graph when the relationship is one-way (follows, prerequisites, one-way streets).

4 SPECIAL GRAPHS

Graph	Edges	Degrees
complete K_n	$\binom{n}{2} = \frac{n(n-1)}{2}$	all $n-1$
cycle C_n	n	all 2
complete bipartite $K_{m,n}$	mn	n and m
tree on n vertices	$n-1$	at least 2 leaves

Degree sequence test (simple graphs): even sum, no degree above $n-1$, then Havel–Hakimi. $4, 2, 1, 1$ fails (degree 4 with only 3 other vertices).

5 GRAPH COLORING



Adjacent vertices get different colors. $\chi(C_5) = 3$ (odd cycle), $\chi(C_6) = 2$, $\chi(K_n) = n$, bipartite $\chi = 2$. Scheduling: vertices = events, edges = conflicts.

6 COMMON MISTAKES

✓ Do remember

- A loop adds 2 to the degree.
- State whether the graph is directed and whether loops are allowed.
- A lower bound (a triangle) plus a coloring proves χ.

✘ Common mistakes

- Adding degrees and calling it the number of edges: divide by 2
- “Every cycle needs 3 colors”: only odd cycles
- Treating an upper bound as χ: $\chi \le \Delta + 1$ is a bound

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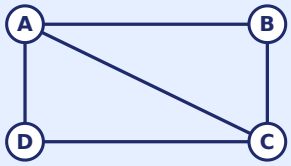
More practice: walks, trails, paths, circuits, Euler and Hamiltonian circuits, trees and spanning trees

1 WALKS, TRAILS, PATHS, CIRCUITS

Term	Repeats an edge?	Repeats a vertex?
walk	allowed	allowed
trail	**no**	allowed
path	no	**no**
circuit (closed trail)	no	allowed (ends at start)
cycle (simple circuit)	no	only start = end

In the square A–B–C–D–A with diagonal AC: A–B–C–A–D is a trail, not a path; A–B–C–D is a path.

2 EULER TRAILS AND CIRCUITS

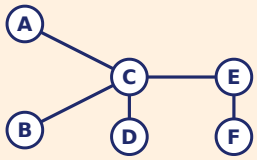


Uses **every edge** once. Connected graph: all degrees even ⟹ Euler circuit; exactly two odd vertices ⟹ Euler trail from one to the other; otherwise neither. Here A, C have degree 3: trail A–B–C–A–D–C.

3 HAMILTONIAN CYCLES

Visits every **vertex** once and returns. No simple degree test decides it: two triangles sharing a vertex have all degrees even (Euler circuit) but no Hamiltonian cycle. K_n has $\frac{(n-1)!}{2}$ Hamiltonian cycles (12 for K_5).

4 TREES



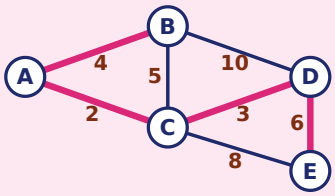
A **tree** is connected with no cycles. With n vertices it has $n-1$ edges; every edge is a bridge; there are at least 2 leaves (here A, B, D, F). Degree sum $2(n-1)$ helps count leaves.

5 MINIMUM SPANNING TREE

Kruskal: take edges from lightest to heaviest, skipping any that would make a cycle.

Example: AB 4, AC 2, BC 5, BD 10, CD 3, CE 8, DE 6
Take: AC 2, CD 3, AB 4, DE 6
Skip: BC 5 (cycle A–B–C)

Answer: weight 15



6 MISTAKES AND CHALLENGE

✓ Do remember

- Euler = every edge; Hamilton = every vertex.
- An Euler trail must start at an odd vertex.
- A connected graph with $n-1$ edges is a tree.

✗ Common mistakes

- "Most degrees are even, so there is an Euler circuit": all must be even
- Including a cycle-forming edge in Kruskal: skip it
- Challenge: Königsberg (degrees 5, 3, 3, 3): four odd vertices: no Euler trail